ON MONOMIAL IDEAL RINGS AND A THEOREM OF TREVISAN

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Abstract. A direct proof is presented of a form of Alvise Trevisan's result [7], that every monomial ideal ring is represented by the cohomology of a topological space. Certain of these rings are shown to be realized by polyhedral products indexed by simplicial complexes.

1. Introduction

In the paper [7], Alvise Trevisan showed that every ring which is a quotient of an integral polynomial ring by an ideal of monomial relations, can be realized as the integral cohomology ring of a topological space. Moreover, he showed that the rings could be all realized with spaces which are generalized Davis-Januszkiewicz spaces. These spaces are colimits over *multicomplexes* which are generalizations of simplicial complexes.

Here is presented a direct proof of the "realization" part of Trevisan's theorem. It uses a result of Fröberg from [5] which asserts that a map known as "polarization" produces in a natural way, a regular sequence of degree-two elements. This allows for the realization of any monomial ideal ring by a certain pullback.

It is noted also that certain families of monomial ideal rings, beyond Stanley-Reisner rings, can be realized as generalized Davis-Januszkiewicz spaces based on ordinary simplicial complexes. Of course, as Trevisan shows, multicomplexes are needed in general.

Through the paper, all cohomology is taken with *integral* coefficients.

2. The main result

Let $\mathbb{Z}[x_1, \ldots, x_n]$ be a polynomial ring and

$$M = \{m_j\}_{j=1}^{r}, \qquad m_j = x_1^{t_{1j}} x_2^{t_{2j}} \cdots x_n^{t_{nj}} \tag{2.1}$$

be a set of minimal monomials, that is, no monomial divides another. Here, the exponent t_{ij} might be equal to zero but every x_i must appear in some m_j. Notice that the set M is determined by the $n \times r$ matrix (t_{ij}). Denote by $I(M)$ the ideal in $\mathbb{Z}[x_1, \ldots, x_n]$ generated by the minimal monomials m_j and set

$$A = A(M) = \mathbb{Z}[x_1, \ldots, x_n]/I(M) \tag{2.2}$$

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a *monomial ideal ring*. From this is defined a second monomial ideal ring $A(\overline{M})$ with monomial ideal generated by square free monomials. For each $i = 1, 2, \ldots, n$ set

$$t_i = \max\{t_{i1}, t_{i2}, \ldots, t_{ir}\} \tag{2.3}$$

the largest entry in the i-th row of (t_{ij}). Next, introduce new variables $y_{i1}, y_{i2}, \ldots, y_{it_i}$ for each $i = 1, 2, \ldots, n$. For each monomial $m_j = x_1^{t_{1j}} x_2^{t_{2j}} \cdots x_n^{t_{nj}}$, set

$$\overline{m}_j = (y_{11}y_{12}\cdots y_{1t_{1j}})(y_{21}y_{22}\cdots y_{2t_{2j}})\ \cdots\ (y_{n1}y_{n2}\cdots y_{nt_{nj}}). \tag{2.4}$$

Let $\overline{M} = \{\overline{m}_j\}_{j=1}^r$ and define an algebra B by

$$B = B(\overline{M}) = \mathbb{Z}[y_{11}, y_{12}, \ldots, y_{1t_1}, y_{21}, y_{22}, \ldots, y_{2t_2},\ \ldots\ , y_{n1}, y_{n2}, \ldots, y_{nt_n}]/I(\overline{M}). \tag{2.5}$$

The monomials here are square-free so B is a Stanley-Reisner algebra which determines a simplicial complex $K(\overline{M})$. (This process which constructs B from A is known in the literature as *polarization*.) Associated to this simplicial complex is a fibration

$$Z\big(K(\overline{M}); (D^2, S^1)\big) \longrightarrow \mathcal{DJ}(K(\overline{M})) \longrightarrow BT^{d(\overline{M})}$$

where $d(\overline{M}) = \sum_{1=1}^n t_i$, with t_i as in (2.3), $\mathcal{DJ}(K(\overline{M}))$ is the Davis-Januszkiewicz space of the simplicial complex $K(\overline{M})$ and $Z\big(K(\overline{M}); (D^2, S^1)\big)$ is the moment-angle complex corresponding to $K(\overline{M})$, ([3]). Recall that the Davis-Januszkiewicz space has the property that

$$H^*\big(\mathcal{DJ}(K(\overline{M}))\big) \cong B. \tag{2.6}$$

Define next a diagonal map $\Delta\colon T^n \longrightarrow T^{d(\overline{M})}$ by

$$\Delta(x_1, x_2, \ldots, x_l) = \big(\Delta_{t_1}(x_1), \Delta_{t_2}(x_2), \ldots, \Delta_{t_n}(x_l)\big) \tag{2.7}$$

where $\Delta_{t_i}(x_i) = (x_i, x_i, \ldots, x_i) \in T^{t_i}$. In the diagram below, let $W(A)$ be defined as the pullback of the fibration.

$$\begin{tikzcd} Z\big(K(\overline{M}); (D^2, S^1)\big) \arrow[r, "="'] \arrow[d] & Z\big(K(\overline{M}); (D^2, S^1)\big) \arrow[d] \\ W(A) \arrow[r, "\widetilde{\Delta}"'] \arrow[d] & \mathcal{DJ}(K(\overline{M})) \arrow[d] \\ BT^n \arrow[r, "B\Delta"] & BT^{d(\overline{M})} \end{tikzcd} \tag{2.8}$$

The diagram (2.8) extends to to the right and produces the fibration

$$W(A) \xrightarrow{\widetilde{\Delta}} \mathcal{DJ}(K(\overline{M})) \xrightarrow{p} BT^{d(\overline{M})-n}. \tag{2.9}$$

Recall that $d(\overline{M}) = \sum_{1=1}^{n} t_i$ and choose generators

$$H^*(BT^{d(\overline{M})-n}) \;\cong\; \mathbb{Z}[u_{12}, \ldots, u_{1t_1}, u_{22}, \ldots, u_{2t_2},\; \ldots\; , u_{n2}, \ldots, u_{nt_n}],$$

so that

$$p^*(u_{ik_i}) \;=\; y_{i1} - y_{ik_i} \quad i = 1, 2, \ldots, n, \;\; k_i = 2, 3, \ldots t_i.$$

The fact that p^* is determined by the diagonal Δ in diagram (2.8), allows this choice. Set $\theta_{ik_i} := p^*(u_{ik_i})$. The proposition following is a basic result about the diagonal map Δ, (the *polarization* map); a proof may be found in [5, page 30].

Proposition 2.1 (Fröberg)**.** *Over any field k, the sequence $\{\theta_{ik_i}\}$ is a regular sequence of degree-two elements in the ring $H^*\big(\mathcal{D}\mathcal{J}(K(\overline{M})); k\big)$.*

This result allows for a direct proof of the realization theorem.

Theorem 2.2. *There is an isomorphism of rings*

$$H^*\big(W(A); \mathbb{Z}\big) \longrightarrow A(M).$$

Proof. Working over a field k and following Masuda-Panov, [6, Lemma 2.1], we use the Eilenberg-Moore spectral sequence associated to the fibration (2.9). It has

$$E_2^{*,*} = \mathrm{Tor}^{*,*}_{\mathrm{H}^*(\mathrm{BT}^{\mathrm{d}(\overline{\mathrm{M}})-\mathrm{n}})}\big(\mathrm{H}^*\big(\mathcal{D}\mathcal{J}(\mathrm{K}(\overline{\mathrm{M}})), \mathrm{k}\big).$$

Now $H^*\big(\mathcal{D}\mathcal{J}(K(\overline{M}))\big)$ is free as an $H^*(BT^{d(\overline{M})-n})$-module by Proposition 2.1, so

$$\begin{aligned}
\mathrm{Tor}^{*,*}_{\mathrm{H}^*(\mathrm{BT}^{\mathrm{d}(\overline{\mathrm{M}})-\mathrm{n}})}\big(\mathrm{H}^*\big(\mathcal{D}\mathcal{J}(\mathrm{K}(\overline{\mathrm{M}}))\big), \mathrm{k}\big) &= \mathrm{Tor}^{0,*}_{\mathrm{H}^*(\mathrm{BT}^{\mathrm{d}(\overline{\mathrm{M}})-\mathrm{n}})}\big(\mathrm{H}^*\big(\mathcal{D}\mathcal{J}(\mathrm{K}(\overline{\mathrm{M}}))\big), \mathrm{k}\big)\\
&= H^*\big(\mathcal{D}\mathcal{J}(K(\overline{M}))\big) \otimes_{H^*(BT^{d(\overline{M})-n})} k\\
&= H^*\big(\mathcal{D}\mathcal{J}(K(\overline{M}))\big)/p^*(H^{>0}(BT^{d(\overline{M})-n})).
\end{aligned}$$

It follows that the Eilenberg-Moore spectral sequence collapses at the E_2 term and hence, as groups,

$$H^*\big(W(A)\big) = H^*\big(\mathcal{D}\mathcal{J}(K(\overline{M}))\big)/p^*(H^{>0}(BT^{d(\overline{M})-n}))$$

from which we conclude that $H^*\big(W(A); k\big)$ is concentrated in even degree. Taking $k = \mathbb{Q}$ gives the result that in odd degree, $H^*\big(W(A); \mathbb{Z}\big)$ consists of torsion only. Unless this torsion is zero, the argument above with $k = \mathbb{F}_p$ for an appropriate p, implies a contradiction. It follows that $H^*\big(W(A); \mathbb{Z}\big)$ is concentrated in even degree.

Lemma 2.3. *The integral Serre spectral sequence of the fibration* (2.9) *collapses.*

Proof. The spaces in the fibration have integral cohomology concentrated in even degrees. □

The E_2-term of the Serre spectral sequence is

$$H^*\big(W(A);\mathbb{Z}\big) \otimes H^*\big(BT^{d(\overline{M})-n};\mathbb{Z}\big).$$

It follows that, as a ring, $H^*\big(W(A);\mathbb{Z}\big)$ is the quotient of $H^*\big(\mathcal{D}\mathcal{J}(K(\overline{M}))\big)$ by the two-sided ideal L generated by the image of p^*. So there is an isomorphism of graded rings,

$$H^*\big(W(A);\mathbb{Z}\big) \longrightarrow H^*\big(\mathcal{D}\mathcal{J}(K(\overline{M}))\big)/L \cong A(\overline{M})/L \cong A(M)$$

completing the proof of Theorem 2.2. □

3. On the geometric realization of certain monomial ideal rings by ordinary polyhedral products

In this section, polyhedral products, [1], involving finite and infinite complex projective spaces are used to realize certain classes of monomial ideal rings. As noted earlier, generalizations of the Davis-Januszkiewicz spaces to the realm of multicomplexes are required in order to realize all monomial ideal rings, see Trevisan [7].

The class which can be realized by ordinary polyhedral products is restricted to those monomials

$$M = \big\{m_j\big\}_{j=1}^{r}, \qquad m_j = x_1^{t_{1j}} x_2^{t_{2j}} \cdots x_n^{t_{nj}}$$

of (2.1), which satisfy the condition:

- ⋇ t_{ij} is constant over all monomials m_j which have t_{ij} and *at least one other exponent* both non-zero.

In particular, a monomial ring of the form

$$\mathbb{Z}[x_1,x_2,x_3]/\langle x_1^2x_2, x_1^2x_3^4, x_3^5\rangle \tag{3.1}$$

can be realized by an ordinary polyhedral product. As usual, let $(\underline{X},\underline{A})$ denote a family of CW pairs

$$\{(X_1,A_1),(X_2,A_2),\ldots,(X_n,A_n)\}.$$

Given a monomial ring $A(M)$ of the form (2.2), satisfying the condition ⋇ above, a simplicial complex K and a family of pairs $(\underline{X},\underline{A})$ will be specified so that

$$H^*\big(Z(K;(\underline{X},\underline{A}));\mathbb{Z}\big) = A(M)$$

where $Z(K;(\underline{X},\underline{A})$ represents a polyhedral product as defined in [1].

Construction 3.1. Let K be the simplicial complex on n vertices $\{v_1,v_2,\ldots,v_n\}$ which has a minimal non-face corresponding to each m_i having *at least two* non-zero exponents. If m_i has non-zero exponents

$$t_{j_1 i}, t_{j_2 i}, \ldots, t_{j_t i}$$

then K will have a corresponding minimal non-face $\{v_{j_1}, v_{j_2}, \ldots, v_{j_t}\}$. Moreover, these will be the only minimal non-faces of K.

For example, the ring (3.1) above will have associated to it, the simplicial complex K on vertices $\{v_1, v_2, v_3\}$ and will have minimal non-faces $\{v_1, v_2\}$ and $\{v_1, v_3\}$. So, K will be the disjoint union of a point and a one-simplex.

For the set of monomials M satisfying condition $\circledast$, the cases following are distinguished in terms of (2.1) for fixed $i \in \{1, 2, \ldots, n\}$.

(1) For certain j, $t_{ij} = 1$, $t_{i'j} \neq 0$ for some $i' \neq i$ and $t_{ik} = 0$ otherwise.
(2) For certain j, $t_{ij} = q_i > 1$, $t_{i'j} \neq 0$ for some $i' \neq i$ and $t_{ik} = 0$ otherwise.
(3) $m_j = x_i^{s_i}$ for some j and $t_{ik} = 0$ for $k \neq j$.
(4) $m_j = x_i^{s_i}$ for some j and if $t_{ik} \neq 0$ for $k \neq j$, then $t_{ik} = q_i < s_i$.

With this classification in mind, define a family of CW-pairs

$$(\underline{X}, \underline{A}) = \{(X_i, A_i) \colon i = 1, \ldots, n\}$$

by

$$(X_i, A_i) = \begin{cases} (\mathbb{C}P^\infty, *) & \text{if } i \text{ satisfies (1)}, \\ (\mathbb{C}P^\infty, \mathbb{C}P^{q_i - 1}) & \text{if } i \text{ satisfies (2)}, \\ (\mathbb{C}P^{s_i - 1}, *) & \text{if } i \text{ satisfies (3)}, \\ (\mathbb{C}P^{s_i - 1}, \mathbb{C}P^{q_i - 1}) & \text{if } i \text{ satisfies (4)}. \end{cases} \tag{3.2}$$

The next theorem describes the polyhedral products which have cohomology realizing the monomial ideal rings satisfying condition $\circledast$.

Theorem 3.2. *Let $A(M)$ be a monomial ring of the form (2.2), satisfying the condition $\circledast$ and K, the simplicial complex defined by Construction 3.1, then*

$$H^*\big(Z(K; (\underline{X}, \underline{A})); \mathbb{Z}\big) = A(M)$$

where (X, A) is the pair specified by (3.2).

Remark. The improvement here over [2, Theorem 10.5] consists of the inclusion of cases (3) and (4) above. The polyhedral products which realize the monomial ideal rings discussed in [1] have $X_i = \mathbb{C}P^\infty$ for all $i = 1, 2, \ldots, n$.

Proof of Theorem 3.2. Set $Q = (q_1, q_2, \ldots, q_n)$ with $q_i \geq 1$ for all i and write the spaces A_i of (3.2) as $\mathbb{C}P^{q_i - 1}$ where $q_i = 1$ if $A_i = *$, a point. Write

$$(\underline{X}, \underline{A}) = (\underline{X}, \underline{\mathbb{C}P}^{Q-1}) = \{(X_i, \mathbb{C}P^{q_i - 1}) \colon i = 1.2. \ldots, n\}$$

and consider the commutative diagram

$$
\begin{tikzcd}
H^*(\prod_{i=1}^n X_i) \arrow[d, "i^*"] & H^*(\prod_{i=1}^n \mathbb{C}P^\infty) \arrow[l, "p^*"'] \arrow[d, "k^*"'] \\
H^*\big(Z(K;(\underline{X},\underline{\mathbb{C}P}^{Q-1}))\big) & H^*\big(Z(K;(\underline{\mathbb{C}P}^\infty,\underline{\mathbb{C}P}^{Q-1}))\big) \arrow[l, "h^*"']
\end{tikzcd}
\tag{3.3}
$$

induced by the various inclusion maps. According to [2, Theorem 10.5], there is an isomorphism of rings

$$H^*\big(Z(K;(\underline{\mathbb{C}P}^\infty,\underline{\mathbb{C}P}^{Q-1}))\big) \longrightarrow \mathbb{Z}[x_1,\ldots,x_n]/I(M^Q)$$

where $I(M^Q)$ is the ideal generated by all monomials $x_{i_1}^{q_{i_1}}, x_{i_2}^{q_{i_2}}, \ldots, x_{i_k}^{q_{i_k}}$ corresponding to the minimal non-faces $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ of K. Moreover, the proof of [2, Lemma 10.3] shows that the composition i^*p^* is a surjection. The commutativity of diagram (3.3) implies that these relations all hold in $H^*\big(Z(K;(\underline{X},\underline{\mathbb{C}P}^{Q-1}))\big)$. In addition to these, the relation $x_i^{s_i} = 0$ is included for each i satisfying $X_i = \mathbb{C}P^{s_i-1}$. These relations account for all the relations determined by $I(M)$. The remainder of the argument shows that $I(M)$ determines all relations in $H^*\big(Z(K;(\underline{X},\underline{A}));\mathbb{Z}\big)$. Consider now the space

$$W_k \;=\; \mathbb{C}P^{q_1-1} \times \cdots \times \mathbb{C}P^{q_{k-1}-1} \times X_k \times \mathbb{C}P^{q_{k+1}-1} \times \cdots \times \mathbb{C}P^{q_n-1}$$

corresponding to the simplex $\{v_k\} \in K$, consisting of a single vertex. The composition

$$W_k \longrightarrow Z(K;(\underline{X},\underline{\mathbb{C}P}^{Q-1})) \longrightarrow \textstyle\prod_{i=1}^n X_i$$

factors the natural inclusion $W_k \longrightarrow \prod_{i=1}^n X_i$. From this observation follows the fact that no other monomial relations occur in $H^*\big(Z(K;(\underline{X},\underline{\mathbb{C}P}^{Q-1}))\big)$ other than those determined by $I(M)$. Suppose next that there is a linear relationship of the form

$$a\omega = \sum_{i=1}^{k} a_i\omega_i \tag{3.4}$$

where $a, a_i \in \mathbb{Z}$ and ω, ω_i are monomials in the x_i, $i = 1, 2, \ldots, n$. Without loss of generality, ω and ω_i can be assumed to be not divisible by any of the monomials in M. Suppose $\omega = x_{j_1}^{\lambda_1} x_{j_2}^{\lambda_2} \cdots x_{j_l}^{\lambda_l}$, then $\sigma = \{v_{j_1}, v_{j_2}, \ldots, v_{j_l}\} \in K$ is a simplex and so is a full subcomplex of K. (The corresponding polyhedral product $Z(\sigma;(\underline{X},\underline{\mathbb{C}P}^{Q-1}))$ is a product of finite and infinite complex projective spaces.) This implies, by [4, Lemma 2.2.3], that $H^*\big(Z(\sigma;(\underline{X},\underline{\mathbb{C}P}^{Q-1}))\big)$ must be a direct summand in $H^*\big(Z(K;(\underline{X},\underline{\mathbb{C}P}^{Q-1}))\big)$ contradicting the relation (3.4). $\square$

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